

ITP-SB-94-30
hep-ph/9406367
February, 1994

Some model-independent properties of quark mixing*

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Abstract

We discuss some new invariants of quark mixing and show their usefulness with a simple example. We also present some other new tools for analyzing quark mixing.

* talk presented at the *Yukawa couplings and the origin of mass* workshop, February 1994, University of Florida, Gainesville, Florida

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1 Invariants of quark mixing and their applications

In recent years the increasingly accurate data on quark mixing has stimulated interest in possible predictions concerning quark mass matrices. The parameters of the Cabibbo-Kobayashi-Maskawa mixing matrix V are now known reasonably well [1]. This determination has been made possible partly by the finding that there are only three generations of usual standard-model fermions (with corresponding light or massless neutrinos). Since the diagonalization of the quark matrices in the up and down sectors determines V , one can work back from the knowledge of V to put constraints on the possible forms of (original, nondiagonal) quark mass matrices. However, the data on quark mixing determines these mass matrices only up to an arbitrary unitary similarity transformation. This is a result of the fact that if the up and down quark mass matrices, M_u and M_d , are both acted on by the same unitary operator U_0 according to

$$M_{u,d} \rightarrow U_0 M_{u,d} U_0^\dagger \quad (1)$$

then the mixing matrix V remains unchanged. There have been many attempts to study specific assumed forms for quark matrices. While this is worthwhile, it is desirable to express the constraints from data on V on the quark mass matrices in an invariant form. In Refs. [2, 3] certain invariant functions of the quark mass matrices I_{pq} were introduced, which are expressed in terms of the quark masses squared and the $|V_{ij}|$:

$$I_{pq} = \text{Tr}(H_u^p H_d^q) = \sum_{ij} (m_i^{(u)})^{2p} (m_j^{(d)})^{2q} |V_{ij}|^2 \quad (2)$$

where $H_q = M_q M_q^\dagger$, and $m_i^{(u)}$ and $m_i^{(d)}$ are the masses of quarks in the “up” and “down” charge sectors.

In this paper we introduce some new invariants of the quark mass matrices (with respect to the transformation (1)) which can be expressed in terms of the measurable quantities only. These new invariants help one simplify the algebraic expressions which relate the elements of the quark mass matrices to the data. This important advantage allows for some new uses of the invariants which we will illustrate with some examples.

We introduce the following new invariants of the transformation (1):

$$K_{pq}(\alpha, \beta) = \det(\alpha H_u^p + \beta H_d^q) \quad (3)$$

where $p, q, \alpha, \beta \neq 0$.

The hermitian matrices H_u and H_d can be diagonalized by a unitary similarity transformation:

$$\begin{cases} U_u H_u U_u^\dagger = D_u \\ U_d H_d U_d^\dagger = D_d \end{cases} \quad (4)$$

where $D_q = \text{diag}((m_1^{(q)})^2, (m_2^{(q)})^2, (m_3^{(q)})^2)$ are the diagonal matrices of the quark masses squared.

The mixing matrix V can be written then as:

$$V = U_u U_d^\dagger \quad (5)$$

In order to find an expression for K_{pq} in terms of U_{ij} we will need the following

Theorem

If A and B are two 3×3 matrices such that $\det(A) \neq 0$ and $\det(B) \neq 0$ then the following relation holds:

$$\det(A + B) = \det(A) + \det(B) + \det(A) \text{Tr}(A^{-1}B) + \det(B) \text{Tr}(AB^{-1}) \quad (6)$$

Proof:

We denote the elements of matrices A and B by A_{ij} and B_{ij} correspondingly. Their co-factors (which are equal to the corresponding minors, up to sign) will be written as \hat{A}_{ij} and \hat{B}_{ij} . Then each determinant may be decomposed in a sum (Laplace expansion):

$$\det(A) = \sum_i A_{ij} \hat{A}_{ij} = \sum_j A_{ij} \hat{A}_{ij}$$

$$\det(B) = \sum_i B_{ij} \hat{B}_{ij} = \sum_j B_{ij} \hat{B}_{ij}$$

By definition, the determinant of a 3×3 matrix is a sum of $3! = 6$ terms:

$$\det(A + B) = \sum (-1)^r (A_{1k_1} + B_{1k_1})(A_{2k_2} + B_{2k_2})(A_{3k_3} + B_{3k_3}) \quad (7)$$

where r is the sign of the permutation $\begin{pmatrix} 1 & 2 & 3 \\ k_1 & k_2 & k_3 \end{pmatrix}$.

The terms in the sum (7) which contain only the elements of A can be arranged as $\det(A)$. Similarly, the terms containing only B 's give $\det(B)$. The terms containing one element of A multiplied by two elements of B , or visa versa, can be rewritten as:

$$\sum_{i,j} (A_{ij} \hat{B}_{ij} + B_{ij} \hat{A}_{ij}) \quad (8)$$

We can now use an identity:

$$(A^{-1})_{ij} = \frac{1}{\det(A)} \hat{A}_{ji}$$

to rewrite (8) as:

$$\sum_{i,j} (A_{ij} \hat{B}_{ij} + B_{ij} \hat{A}_{ij}) = \det(B) \sum_{ij} A_{ij} (B^{-1})_{ji} + \det(A) \sum_{ij} (A^{-1})_{ji} B_{ij} =$$

$$\det(A) \operatorname{Tr}(A^{-1}B) + \det(B) \operatorname{Tr}(AB^{-1})$$

Altogether we get

$$\det(A + B) = \det(A) + \det(B) + \det(A) \operatorname{Tr}(A^{-1}B) + \det(B) \operatorname{Tr}(AB^{-1})$$

which is the statement of the theorem (6). This completes the proof.

Theorem (6) may be easily generalized to the case of 2×2 matrices, in which case the last two terms in (6) are equal and correspond to a redundant counting of the same terms in a sum similar to (7). Thus for the 2×2 matrices we get:

$$\begin{aligned} \det(A + B) &= \det(A) + \det(B) + \det(A) \operatorname{Tr}(A^{-1}B) \equiv \\ &\det(A) + \det(B) + \det(B) \operatorname{Tr}(AB^{-1}) \end{aligned} \quad (9)$$

The immediate consequence of equations (6) and (2) is the following relation:

$$\begin{aligned}
K_{pq}(\alpha, \beta) &\equiv \det(\alpha H_u^p + \beta H_d^q) = \\
&\alpha^3 (m_1^{(u)} m_2^{(u)} m_3^{(u)})^{2p} [1 + (\beta/\alpha) \sum_{ij} [(m_j^{(d)})^{2q}/(m_i^{(u)})^{2p}] U_{ij}] + \\
&\beta^3 (m_1^{(d)} m_2^{(d)} m_3^{(d)})^{2q} [1 + (\alpha/\beta) \sum_{ij} [(m_j^{(u)})^{2p}/(m_i^{(d)})^{2q}] U_{ij}]
\end{aligned} \tag{10}$$

We also notice that

$$\begin{aligned}
K_{pq}(\alpha, \pm\beta) &= \alpha^3 (m_1^{(u)} m_2^{(u)} m_3^{(u)})^{2p} (1 \pm (\beta/\alpha) I_{(-p)q}) \\
&\quad \pm \beta^3 (m_1^{(d)} m_2^{(d)} m_3^{(d)})^{2q} (1 \pm (\alpha/\beta) I_{p(-q)})
\end{aligned} \tag{11}$$

Any four independent invariants from the set $\{I_{pq}, K_{pq}\}$ contain all the physical information about the CKM matrix.

If the mass matrices are assumed to be hermitian, one can introduce a similar set of invariants:

$$\begin{aligned}
\tilde{I}_{pq} &= \text{Tr}(M_u^p M_d^q) \\
\tilde{K}_{pq}(\alpha, \beta) &= \det(\alpha M_u^p + \beta M_d^q)
\end{aligned} \tag{12}$$

The formulae, similar to (2), (10) and (11), will also hold for \tilde{I}, \tilde{K} :

$$\tilde{I}_{pq} = \text{Tr}(M_u^p M_d^q) = \sum_{ij} (m_i^{(u)})^p (m_j^{(d)})^q |V_{ij}|^2 \tag{13}$$

$$\begin{aligned}
\tilde{K}_{pq}(\alpha, \beta) &\equiv \det(\alpha M_u^p + \beta M_d^q) = \\
&\alpha^3 (m_1^{(u)} m_2^{(u)} m_3^{(u)})^p [1 + (\beta/\alpha) \sum_{ij} [(m_j^{(d)})^q/(m_i^{(u)})^p] U_{ij}] + \\
&\beta^3 (m_1^{(d)} m_2^{(d)} m_3^{(d)})^q [1 + (\alpha/\beta) \sum_{ij} [(m_j^{(u)})^p/(m_i^{(d)})^q] U_{ij}]
\end{aligned} \tag{14}$$

The odd powers of the mass eigenvalues may appear in some of the \tilde{I} and \tilde{K} -type invariants. As usual, the signs of the fermion masses are ambiguous. However, for a given model, different choices of signs will in general result in different predictions for the CKM

matrix. This is because in general the mass matrices do not commute with all of the diagonal matrices of the form $diag(\pm 1, \pm 1, \pm 1)$, where the $+$ and $-$ signs are chosen arbitrarily but so as to not get the plus or minus identity. We discussed the effect of the sign choices on the CKM mixing parameters on the example of some particular model [4]. Once the choice of signs for the fermion eigenvalues (determined in practice by the best fit to the data) is made, there is no further sign-related ambiguity in the \tilde{I} and \tilde{K} -type invariants.

Now we would like to illustrate the usefulness of the invariants discussed above. As an example, we will take the model proposed in [4]. This model gives predictions for the low energy data on fermion masses and mixing which are in reasonable agreement with experiment. This model is formulated in the context of an $SO(10)$ supersymmetric grand unified theory. The model has the following Yukawa matrices at the GUT scale:

$$M_u = \begin{pmatrix} 0 & A_u & 0 \\ A_u & B_u & 0 \\ 0 & 0 & C_u \end{pmatrix} \quad (15)$$

$$M_d = \begin{pmatrix} 0 & A_d e^{i\phi} & 0 \\ A_d e^{-i\phi} & B_d e^{i\theta} & B_d \\ 0 & B_d & C_d \end{pmatrix} \quad (16)$$

$$M_e = \begin{pmatrix} 0 & A_d e^{i\phi} & 0 \\ A_d e^{-i\phi} & -3B_d e^{i\theta} & -3B_d \\ 0 & -3B_d & C_d \end{pmatrix} \quad (17)$$

The low energy effective theory below GUT thresholds is assumed to be the Minimal Supersymmetric Standard Model (MSSM). After the renormalization effects are taken into account, the model agrees with the data on fermion mixing for a broad range of the t -quark mass, $m_t = 150...190 \text{ GeV}$. In these fits the phase θ is relatively unimportant and can be taken to be zero.

The values of the parameters $|A_q|$ and C_q ($q = u, d$), in (15) and (16) are simply related to the masses of quarks. On the contrary, the phase ϕ , which must have a nonzero value in order for the model to agree with experiment, depends on both the masses of quarks and their mixing parameters. It is a common procedure to diagonalize the matrices M_u and M_d numerically (or by means of the Taylor series expansion in terms of the quark mass ratios)

and compare the corresponding CKM matrix to the data. On the other hand, the method of invariants discussed above offers a simpler and more elegant solution: the phase ϕ can be expressed analytically in terms of the measurable quantities only. Because the mass matrices in (15) and (16) are hermitian for $\theta = 0$, one can use a \tilde{K} -type invariant. On one hand, the allowed range for the value of $\tilde{K}_{11}(1, 1)$ is known in terms of the measurable quantities from (14). On the other hand,

$$\begin{aligned}\tilde{K}_{11}(1, 1) &= \det(M_u + M_d) = \\ &= |A_u + A_d| (C_u + C_d) = \\ &= (2|A_u A_d| \cos(\phi) + |A_u|^2 + |A_d|^2)(C_u + C_d)\end{aligned}\tag{18}$$

And therefore

$$\cos(\phi) = \frac{1}{2|A_u||A_d|} (\tilde{K}_{11}(1, 1)/(C_u + C_d) - |A_u|^2 - |A_d|^2)\tag{19}$$

The right-hand side of (19) is known in terms of the quark masses and the CKM mixing parameters. Therefore, one can use the relation (19) to evaluate the phase ϕ without having to explicitly diagonalize the mass matrices.

For this model and others, it is important to have a general methodology to determine (1) how many unremovable phases there are in quark mass matrices and (2) which elements of these matrices can be rephased to be real. We have presented a general method to answer both of these questions [5]. The analogous questions for lepton mass matrices are more complicated, but we have also given a general answer to them [6].

In summary, we have introduced and studied some new invariants of the quark mass matrices which are model- and weak basis-independent. The identities (10) and (2), as well as (13) and (14), involving these invariants, provide important constraints on the possible forms of quark mass matrices M_u and M_d since they directly relate the elements of these matrices to the measurable parameters $|V_{ij}|^2$ and quark masses, and thereby enable one to avoid the explicit calculation of the eigenvectors of H_u and H_d .

2 A method to generate families of viable quark mass matrices

In this section we would like to present some useful mathematical tools, first proposed in [7], for the study of quark mass matrices and the connection with quark mixing. Our method was recently applied in [8] to generate the families of acceptable solutions for the fermion mass matrices.

The idea of the method is to utilize the well-known experimental fact that the CKM matrix is close to the 3×3 identity matrix: $|V_{ij}| \approx \delta_{ij}$. We make use of this fact by writing V as

$$V = e^{i\alpha H} \quad (20)$$

where H is some hermitian matrix and α is a real number. One may choose H to have its dominant (largest, in absolute value) eigenvalue to be 1. Then for α consistent with the data [1], we find that $|\alpha| \approx 0.3$. We can now expand V in the powers of α :

$$V = \mathbf{1} + i\alpha H - \frac{1}{2}\alpha^2 H^2 + \dots + \frac{1}{n!}(i\alpha H)^n + \dots \quad (21)$$

It was shown in Ref. [7] that for any practical purposes it is sufficient to consider the first and the second order in α to match the precision to which the relevant quantities (CKM parameters, quark masses, etc.) are known.

The matrix H , corresponding to a given matrix V with distinct eigenvalues v_i , can be easily computed using the Sylvester's theorem:

$$i\alpha H = \sum_{k=1}^3 \ln(v_k) \frac{\prod_{i \neq k} (V - v_i \times \mathbf{1})}{\prod_{i \neq k} (v_k - v_i)} \quad (22)$$

Then the usual system of equations:

$$\begin{cases} U_{u,L} M_u U_{u,R}^\dagger = \text{diag}(m_u, m_c, m_t) \equiv D_u \\ U_{d,L} M_d U_{d,R}^\dagger = \text{diag}(m_d, m_s, m_b) \equiv D_d \\ U_{u,L} U_{d,L}^\dagger = V \end{cases} \quad (23)$$

is satisfied for the family of solutions:

$$\left\{ \begin{array}{l} M_u = U_u^\dagger D_u U_u = \\ D_u + i\alpha x [D_u, H] - \frac{1}{2}\alpha^2 x^2 [[D_u, H], H] + \dots \\ \\ M_d = U_d^\dagger D_d U_d = \\ D_d + i\alpha(x-1) [D_d, H] - \frac{1}{2}\alpha^2 (x-1)^2 [[D_d, H], H] + \dots \end{array} \right. \quad (24)$$

depending on some arbitrary parameter x . (It is assumed that $|x|$, as well as $|1-x|$, is sufficiently small to preserve the convergence of the series.)

To summarize, in this presentation we have discussed two different approaches to analyzing quark mixing. First, we introduced some new invariants of quark mixing and showed the usefulness of our method of invariants on a simple example. Then we have also discussed some new mathematical tools which allow one to generate the mass matrices consistent with the data.

The author would like to thank Professor Robert Shrock for many helpful discussions and comments.

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